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Chapter 1: Probability Theory

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Objectives of the chapter

1. Introduce basic concept of sets
2. Introduce basic concepts of probability
3. Introduce some useful counting methods

CLO1	Explain basic concepts probability, joint probability, conditional probability, independence, total probability, and Bayes' rule.
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1. Set definitions

- A set can be defined as a collection of objects. Sets are generally denoted by capital letters as: A, B, C, ...
- The individual objects forming the set are called "elements" or "members". They are generally denoted by lower case letters as: a, b, c, ...

- If an element g belongs to a set G , we write:

$$g \in G \quad (1)$$

Otherwise, we say g is not a member of G , we write:

$$g \notin G \quad (2)$$

- A set is specified by the content of two braces: $\{\cdot\}$.
- **Representation of sets:**
 - Tabular method: the elements are enumerated explicitly. For example: $A = \{3, 4, 5, 6\}$.
 - Rule method: the content of the set is specified using a rule. This representation is more convenient when the set is large. For example:

$$G = \{g \mid g \text{ is an integer and } 3 \leq g \leq 6\} \quad (3)$$



- **Countable and uncountable sets:** A set is called to be "countable" if its elements can be put in one-to-one correspondence with the integers 1, 2, ... etc. Otherwise, it is called "uncountable".
- **Empty set:** A set G is said to be empty, if it has no elements. It is also called null set and it is denoted by \emptyset .
- **Finite and infinite sets:** A finite set is either empty set or has elements that can be counted, with the counting process terminating. If a set is not finite it is called infinite.

- **Subset:** Given two sets A and B , if every element of A is also an element of B , A is said to be contained in B . A is known as a subset of B . We write:

$$A \subseteq B \quad (4)$$


- **Proper subset:** If at least one element in B is not in A , then A is a proper subset of B , denoted by

$$A \subset B \quad (5)$$

- **Disjoint sets:** If two sets A and B have no common elements, then they are called disjoint or mutually exclusive.

Example 1:

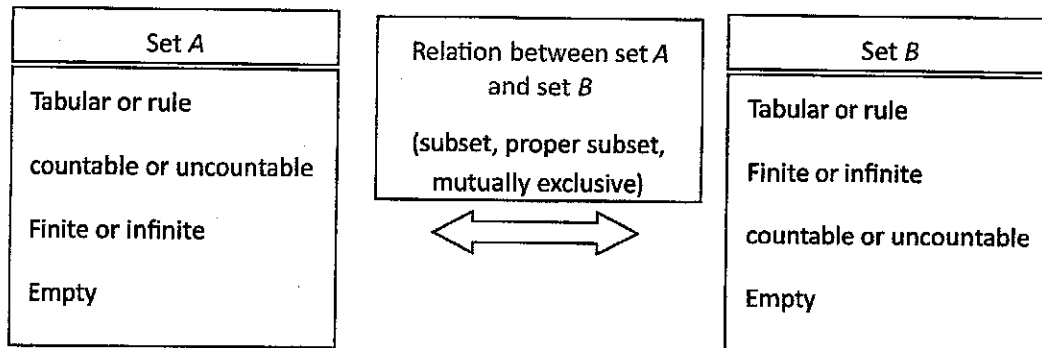
Let us consider the following four sets:

$$A = \{1,3,5,7\} \quad D = \{0\}$$

$$B = \{1,2,3, \dots\} \quad E = \{2,4,6,8,10,12,14\}$$

$$C = \{c | c \text{ is real and } 0.5 < c \leq 8.5\} \quad F = \{f | f \text{ is real and } -5 < f \leq 12\}$$

Illustrate the previous concepts using the sets A, B, C, D, E, F .

**Solution:**

- The set A is tabularly specified, countable, and finite.
 - Set A is contained in sets B, C and F.
 - The set B is tabularly specified and countable, but is infinite.
 - Set C is rule-specified, uncountable, and infinite.
 - Sets D and E are countably finite.
 - Set F is uncountably infinite.
 - $C \subset F, D \subset F, E \subset B$.
 - Sets B and F are not sub sets of any of the other sets or of each other.
 - Sets A, D and E are mutually exclusive of each other.
- **Universal set:** The set of all elements under consideration is called the universal set, denoted S . All sets (of the situation considered) are subsets of S .
 If we have a set S with n elements, then there are 2^n subsets.
 In case of rolling die, the universal set is $S = \{1,2,3,4,5,6\}$ and the number of subsets is $2^6 = 64$ subsets.

Example 2:

Determine the subsets of the following universal set $S = \{1,2,3,4\}$

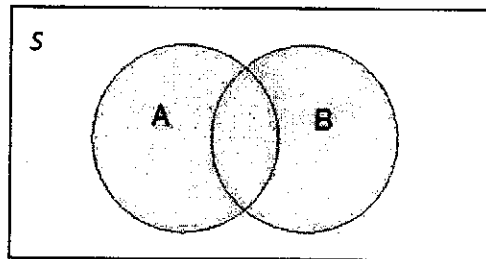
Solution:

The universal set is $S = \{1, 2, 3, 4\}$ and the number of subsets is $2^4=16$ subsets.

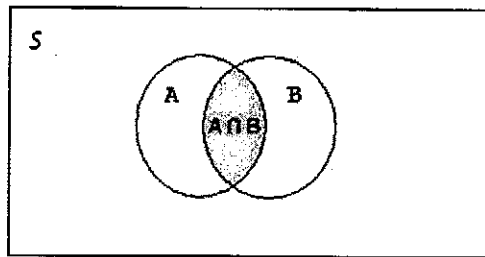
1	\emptyset	9	$\{2,3\}$
2	$\{1\}$	10	$\{2,4\}$
3	$\{2\}$	11	$\{3,4\}$
4	$\{3\}$	12	$\{1,2,3\}$
5	$\{4\}$	13	$\{1, 3,4\}$
6	$\{1,2\}$	14	$\{1,2,4\}$
7	$\{1,3\}$	15	$\{2,3,4\}$
8	$\{1,4\}$	16	$\{1,2,3,4\}$

2. Set Operations

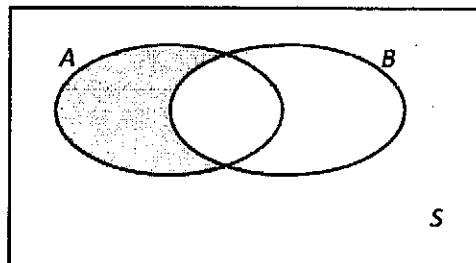
- **Venn diagram:** is a graphical representation of sets to help visualize sets and their operations.
- **Union:** set of all elements that are members of A B or or both and is denoted by $A \cup B$.



- **Intersection:** set of all elements which belong to both A and B and is denoted by $A \cap B$

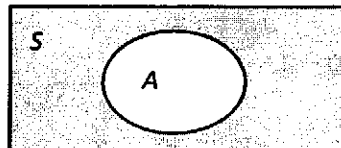


- **Difference:** Set consisting of all elements in A which are not in B and is denoted as $A - B$



- **Complement:** The set composed of all members in S and not in A is the complement of A and denoted \bar{A} . Thus

$$\bar{A} = S - A \tag{6}$$



It is easy to see that $\overline{\emptyset} = S$, $\overline{S} = \emptyset$, $A \cup \overline{A} = S$, and $A \cap \overline{A} = \emptyset$

Example 3:

Let us illustrate these concepts on the following four sets

$$S = \{a \mid a \text{ is an integer and } 1 \leq a \leq 12\}$$

$$A = \{1, 3, 5, 12\}$$

$$B = \{2, 6, 7, 8, 9, 10, 11\}$$

$$C = \{1, 3, 4, 6, 7, 8\}$$

Solution:

▪ **Unions and intersections**

$$A \cup B = \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12\} \quad A \cap B = \emptyset$$

$$A \cup C = \{1, 3, 4, 5, 6, 7, 8, 12\} \quad A \cap C = \{1, 3\} \quad A$$

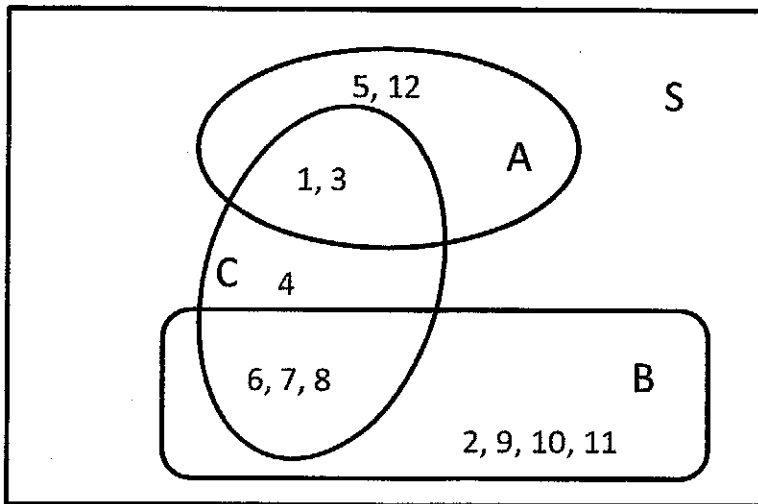
$$B \cup C = \{1, 2, 3, 4, 6, 7, 8, 9, 10, 11\} \quad B \cap C = \{6, 7, 8\}$$

▪ **Complements**

$$\overline{A} = \{2, 4, 6, 7, 8, 9, 10, 11\}$$

$$\overline{B} = \{1, 3, 4, 5, 12\}$$

$$\overline{C} = \{2, 5, 9, 10, 11, 12\}$$



- Algebra of sets:

- ✓ Commutative law: $A \cap B = B \cap A$

$$A \cup B = B \cup A$$

- ✓ Distributive law: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- ✓ Associative law: $(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C$

$$(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$$

- ✓ De Morgan's Law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

3. Probability

- We use probability theory to develop a mathematical model of an experiment and to predict the outcome of an experiment of interest.
- A single performance of the experiment is called a **trial** for which there is an **outcome**.
- In building the relation between the set theory and the notion of probability, we call the set of all possible distinct outcomes of interest in a particular experiment as the **sample space S**
 - The sample space S may be different for different experiments.
- The sample space S can be discrete or continuous, countable or uncountable, finite or infinite.
- An **event** is a particular outcome or a combination of outcomes.
- An event is a subset of the sample space S.

Probability definition and axioms

- Let A an event defined on the sample space S . The probability of the event A denoted as $P(A)$ is a function that assigns to A a real number such that:

✓ **Axiom1:** $P(A) \geq 0$ (7)

✓ **Axiom2:** $P(S) = 1$ (8)



- ✓ **Axiom3:** if we have N events $A_n, n = 1, 2, \dots, N$ defined on the sample space S , and having the propriety: $A_m \cap A_n = \emptyset$ for $m \neq n$ (mutually exclusive events). Then:

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n) \quad (9)$$

Or $P(\cup_{n=1}^N A_n) = \sum_{n=1}^N P(A_n)$ (10)

Some Properties:

- For every event A , its probability is between 0 and 1:

$$0 \leq P(A) \leq 1 \quad (11)$$

- The probability of the impossible event is zero

$$P(\emptyset) = 0 \quad (12)$$

- If \bar{A} is the complement of A , then:

$$P(\bar{A}) = 1 - P(A) \quad (13)$$

□ To model a real experiment mathematically, we shall :

- Define the sample space.
- Define the events of interest.
- Assign probabilities to the events that satisfy the probability axioms.

Example 4:

An experiment consists of observing the sum of two six sided dice when thrown randomly.

Develop a model for the experiment.

- Determine sample space S
- Let the event A be: "the sum events is 7"
- Let the event B be: " $8 < \text{sum} \leq 11$ ".

Determine $P(A), P(B), P(\bar{A}), P(\bar{B})$.

Solution

The sample space: if one experiments can result in any of m possible outcomes and if another experiment can result in any of n possible outcomes, then there are nm possible outcomes of the two experiments (basic principle of counting). The sample space consists of $6^2 = 36$ different outcomes.

$$S = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}$$

Let $B = \{8 < \text{sum} \leq 11\}$. events $A = \{\text{sum} = 7\}$,

In probability assignment, if the dice are not biased, then $P(\text{each outcome}) = 1/36$.

To obtain $p(A)$ and $P(B)$, note that the outcomes are mutually exclusive: therefore, axiom 3 applies:

$$P(A) = P\left(\bigcup_{i=1}^6 S_{i,7-i}\right) = 6\left(\frac{1}{36}\right) = \frac{1}{6}$$

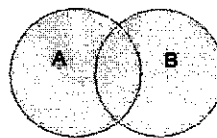
$$P(B) = 9\left(\frac{1}{36}\right) = \frac{1}{4}$$

4. Joint and conditional probability

Joint probability

- When two events A and B have some elements in common (not mutually exclusive), then axiom 3 cannot be applied.
- The probability $P(A \cap B)$ is called the joint probability for the events A and B which intersect in sample space.

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$



(14)

Equivalently:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Conditional probability

- Given some event B with nonzero probability $P(B) > 0$
- We defined, the conditional probability of an event A given B , by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad (15)$$

- $P(A|B)$ is the probability that A will occur given that B has occurred.
- If the occurrence of event B has no effect on A , we say that A and B are independent events.

In this case,

$$P(A|B) = P(A) \quad (16)$$

Which means that:

$$P(A \cap B) = P(A)P(B) \quad (17)$$

Example 4

In a box there are 100 resistors having resistance and tolerance as shown below:

	Tolerance		
Resistance(Ω)	5%	10%	Total
22	10	14	24
47	28	16	44
100	24	8	32

Total	62	38	100
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Let a resistor be selected from the box and define the events:

A = 'Draw 47 Ω resistor'

B = 'Draw resistor with 5% tolerance'

C = 'Draw 100 Ω resistor'

Find $P(A)$, $P(B)$, $P(C)$, $P(A \cap B)$, $P(A \cap C)$, $P(B \cap C)$, $P(A|B)$, $P(A|C)$, $P(B|C)$.

Solution

$$P(A)=P(47\ \Omega)=44/100=0.44. \quad P(B)=P(5\%)=62/100=0.62 \quad P(C)=P(100\ \Omega)=32/100=0.32$$

Joint probabilities are:

$$P(A \cap B) = P(47\ \Omega \cap 5\%) = \frac{28}{100} = 0.28$$

$$P(A \cap C) = P(47\ \Omega \cap 100\ \Omega) = 0$$

$$P(B \cap C) = P(5\% \cap 100\ \Omega) = \frac{24}{100} = 0.24$$

The conditional probabilities become:

$P(A/B) = P(47\ \Omega / 5\%)$ is the probability of drawing a 47 Ω resistor given that the resistor drawn is 5%.

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{28}{62}$$

$$P(A/C) = \frac{P(A \cap C)}{P(C)} = 0$$

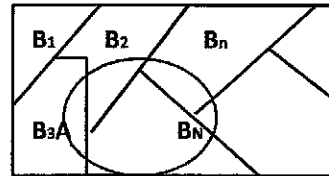
$$P(B/C) = \frac{P(B \cap C)}{P(C)} = \frac{24}{32}$$

Total probability

- Suppose we are given n mutually exclusive events $B_n, n= 1, \dots, N$ such that:

$$\bigcup_{n=1}^N B_n = S \quad (18)$$

and



$$B_m \cap B_n = \emptyset \text{ for } m \neq n$$

- The **total probability** of an event A defined on the sample space S can be expressed in terms of conditional probabilities as follows:

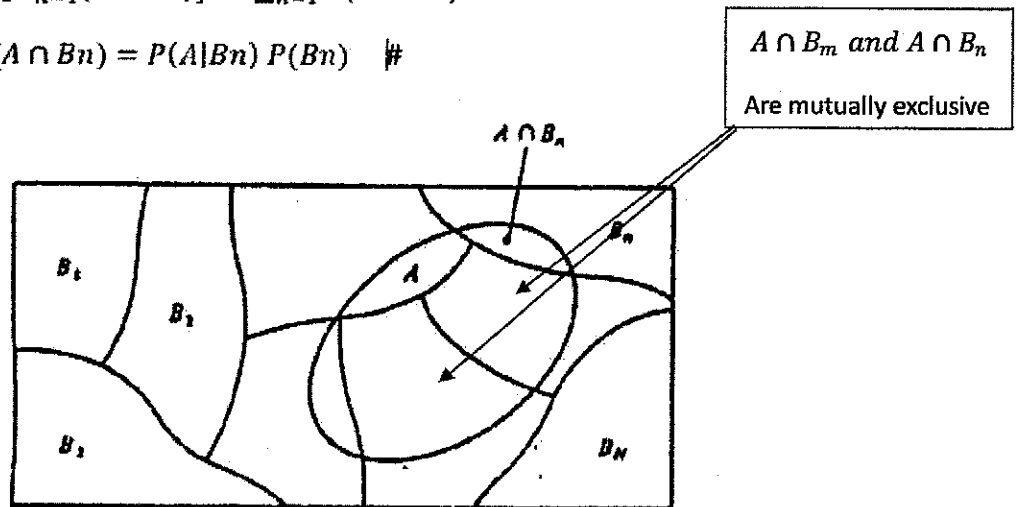
$$P(A) = \sum_{n=1}^N P(A|B_n)P(B_n) \quad (19)$$

Prove: since $A = A \cap S = A \cap (\bigcup_{n=1}^N B_n) = \bigcup_{n=1}^N (A \cap B_n)$

As shown in the diagram, $A \cap B_n$ events are mutually exclusive; therefore:

$$P(A) = P[\bigcup_{n=1}^N (A \cap B_n)] = \sum_{n=1}^N P(A \cap B_n)$$

Since $P(A \cap B_n) = P(A|B_n) P(B_n)$ #



Bayes' Theorem:

- The Bayes rule expresses a conditional probability in terms of other conditional probabilities, we have:

$$P(B_n|A) = \frac{P(B_n \cap A)}{P(A)} \quad (20)$$

$$P(A|B_n) = \frac{P(A \cap B_n)}{P(B_n)} \quad (21)$$

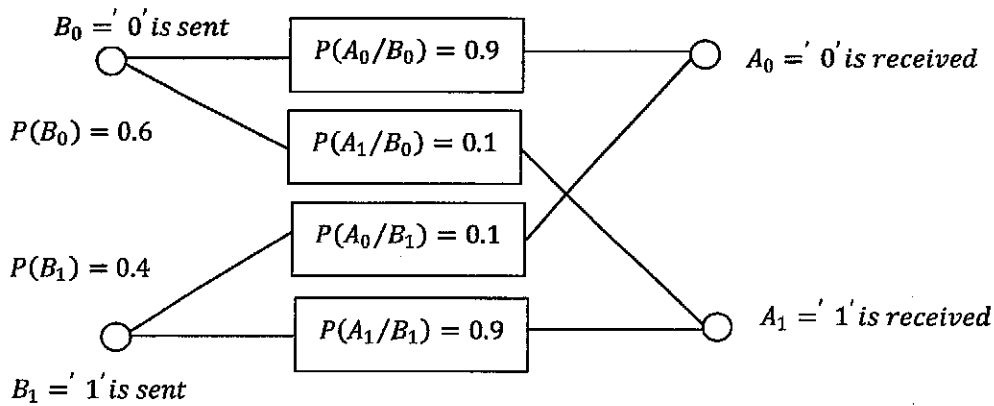
Therefore one form of the Bayes theorem is given by equating these two expressions:

$$P(B_n|A) = \frac{P(A|B_n)P(B_n)}{P(A)} \quad (22)$$

which can be written also as another form:

$$P(B_n|A) = \frac{P(A|B_n)P(B_n)}{P(A|B_1)P(B_1)+\dots+P(A|B_N)P(B_N)} \quad (23)$$

Example 5: A binary Communication system is described as:



Find:

- $P(A_0)$ ('0' is received).
- $P(A_1)$ ('1' is received).
- $P(B_0/A_0), P(B_0/A_1), P(B_1/A_0), P(B_1/A_1)$.

Solution:

Solu:

$$\text{a) } P(A_0) = P(A_0|B_0)P(B_0) + P(A_0|B_1)P(B_1)$$

$$= 0.9(0.6) + 0.1(0.4) = 0.58$$

$$\text{b) } P(A_1) = P(A_1|B_0)P(B_0) + P(A_1|B_1)P(B_1)$$

$$= 0.1(0.6) + 0.9(0.4) = 0.42$$

Total probability

Note that A_0 and A_1 are mutually exclusive and $P(A_0) + P(A_1) = 1$

$$c) P(B_0 | A_0) = \frac{P(A_0 | B_0) P(B_0)}{P(A_0)} = \frac{0.9(0.6)}{0.58} = 0.931$$

$$P(B_0 | A_1) = \frac{P(A_1 | B_0) P(B_0)}{P(A_1)} = \frac{0.1(0.6)}{0.42} = 0.143$$

$$P(B_1 | A_0) = \frac{P(A_0 | B_1) P(B_1)}{P(A_0)} = \frac{0.1(0.4)}{0.58} = 0.069$$

$$P(B_1 | A_1) = \frac{P(A_1 | B_1) P(B_1)}{P(A_1)} = \frac{0.9(0.4)}{0.42} = 0.857$$

Bayes' Theorem

Note that $P(B_0 | A_1)$ and $P(B_1 | A_0)$ are probabilities of error and $P(B_0 | A_0)$ and $P(B_1 | A_1)$ are probabilities of correct transmission.

5. Independent Events

□ Two events A and B are said to be independent if the occurrence of one event is not affected by the occurrence of the other. That is:

$$P(A|B) = P(A) \quad (24)$$

And we also have

$$P(B|A) = P(B) \quad (25)$$

Since

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A)P(B) \quad (\text{joint occurrence, intersection}) \quad (26)$$

➡ - Note that for mutually exclusive events $P(A \cap B) = 0$

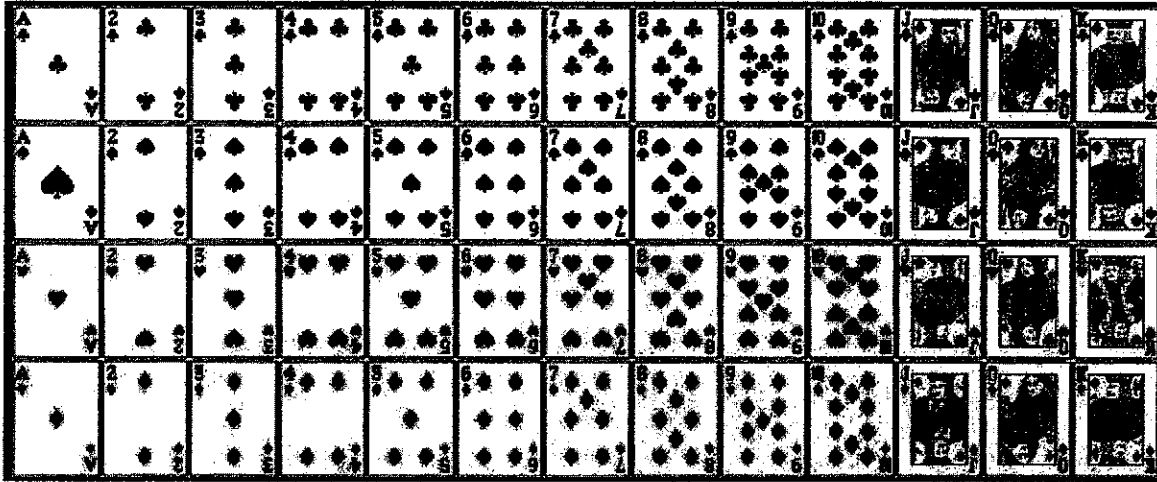
Therefore, for $P(A) \neq 0$, $P(B) \neq 0$, A and B cannot be both mutually exclusive ($A \cap B = \emptyset$), and independent ($A \cap B \neq \emptyset$).

Example 1.5:

One card is selected from 52 card deck. Define events A = “select a king”, B =select Jack or queen” and C = “select a heart”

Find:

- $P(A)$, $P(B)$, $P(C)$
- $P(A \cap B)$, $P(B \cap C)$, $P(A \cap C)$.
- Are the events independent?



Solu:

$$\text{a) } P(A) = \frac{4}{52}, P(B) = \frac{8}{52}, P(C) = \frac{13}{52}$$

It is not possible to simultaneously select a king and a jack or a queen.

$$\text{b) } P(A \cap B) = 0, P(A \cap C) = \frac{1}{52}, P(B \cap C) = \frac{2}{52}$$

We determine whether A , B , and C are independent by pairs.

$$\text{c) } P(A \cap B) = 0 \neq P(A)P(B) \Rightarrow A \text{ and } B \text{ are not independent}$$

$$P(A \cap C) = \frac{1}{52} = P(A)P(C) \Rightarrow A \text{ and } C \text{ are independent}$$

$$P(B \cap C) = \frac{2}{52} = P(B)P(C) \Rightarrow B \text{ and } C \text{ are independent}$$

- In case of multiple events, they are said to be independent if all pairs are independent and:

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

6. Combined Experiments

- A combined experiment consists of forming a single experiment by suitably combining individual experiments.
- These experiments are called *sub-experiments*
- If we have N sample spaces S_n ; $n=1,2,\dots,N$ having elements s_n then the combined sample space is defined as:

$$S=S_1 \times S_2 \times \dots \times S_N \quad (27)$$

Example 6:

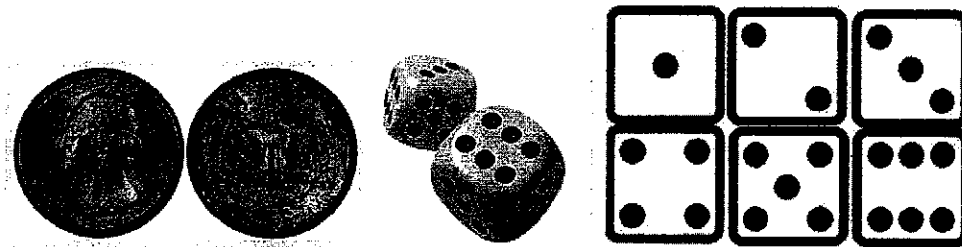
Let us consider the two following sub-experiments:

- Flipping a coin
- Rolling of single die

Determine the sample space S_1 and S_2 corresponding to these two sub-experiments.

Determine the combined sample space S .

Solution:



Solu: for flipping a coin: $S_1=\{H,T\}$

For rolling a die: $S_2=\{1,2,3,4,5,6\}$

$S = S_1 \times S_2 = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$

Example 7:

We flip a coin twice. What is the combined sample space S

Solution:

$$S_1 = \{H, T\}$$

$$S_2 = \{H, T\}$$

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

7. Some counting methods

- Count the number of words of length k with n letters.

Each digit has
3 possibilities

Ex: $n=3$, {A, B, C}, and $k=5$, A B B A C the number $\# = 3^5$ more generally $\# = n^k$.

3 3 3 3 3

- Count the number of words of length k from alphabet of k letters with no allowed repetition (i.e. Permutation of k objects).

Ex: $n=5$ {A, B, C, D, E}, $k=5$, D B E _ _ $\# = k(k-1)(k-2) \dots (2)(1) = k!$
5 4 3 2 1

- Number of words of length k from alphabet with n letters, repetition not allowed (**permutation** ordering is important here)

$$\# = P_k^n = \underset{1}{n} \underset{2}{(n-1)} \underset{3}{(n-2)} \dots \underset{k}{(n-k+1)} = \frac{n!}{(n-k)!}$$

- If order of elements in a sequence is not important (take one element from all the permuted elements), then the number of possible sequences is called **combinations**, which equals P_k^n divided by the number of permutations (orderings) of k elements

$P_k^k = k!$. The number of combinations of k elements taken from n elements $\binom{n}{k}$ is:

$$C_k^n = \binom{n}{k} = \frac{n!}{(n-k)! k!}$$

$\binom{n}{k}$ is called the binomial coefficients.

Example 8:

How many permutations for four cards taken from 52 cards?

Solu: $P_4^{52} = \frac{52!}{(52-4)!} = 52(51)(50)(49) = 6,497,400$

Example 9:

A team of 3 players is to be selected from 5 players, how many teams can be chosen?

Solu: $\binom{5}{3} = \frac{5!}{3! 2!} = 10$

Example 10:

A number is composed of 5 digits. How many way are there for this number?

Solu: no. of ways = $10 \times 10 \times 10 \times 10 \times 10 = 10^5$

8. Bernoulli Trials

- These type of experiments are characterized by two possible outcomes: A and \bar{A} .

For example: flipping a coin, hitting or missing a target, receiving 0 or 1.

- Let $P(A) = p$, Then $P(\bar{A}) = 1-p$.

If the experiment is repeated N times, then the probability that event A occurs k times (regardless of ordering) equals the probability of this sequence multiplied by its number. In this case, \bar{A} will occur $N-k$ times and the probability of this sequence (one sequence) is:

$$P(A) P(A) \dots P(A) P(\bar{A}) P(\bar{A}) \dots P(\bar{A}) = P^k (1 - P)^{N-k}$$

k times *N-k times*

- There are other sequences that will yield k events A and $N-k$ events \bar{A} , From a combinatorial analysis, the number of sequences where A occurs k times in N trials is:

$$\binom{N}{k} = \frac{N!}{k! (N-k)!}$$

Finally we obtain the probability: $P(A \text{ occurs } k \text{ times}) = \binom{N}{k} p^k (1-p)^{N-k}$

Example 12:

A submarine will sink a ship if two or more rockets hit the ship. If the submarine fires 3 rockets and $P(\text{hit}) = 0.4$ for each rocket, what is the probability that the ship will be sunk?

Solution:

$$P(\text{no hits}) = \binom{3}{0} 0.4^0 (1-0.4)^3 = 0.216$$

$$P(1 \text{ hit}) = \binom{3}{1} 0.4^1 (1-0.4)^2 = 0.432$$

$$P(2 \text{ hits}) = \binom{3}{2} 0.4^2 (1-0.4)^1 = 0.288$$

$$P(3 \text{ hits}) = \binom{3}{3} 0.4^3 (1-0.4)^0 = 0.064$$

$$P(\text{ship sunk}) = P(\text{two hits and more}) = P(2 \text{ hits}) + P(3 \text{ hits}) = 0.352$$