1. **Definition of Capacitance**
2. **Calculating Capacitance**
3. **Capacitors with Dielectrics**
4. **Combinations of Capacitors**
5. **Energy Stored in a Charged Capacitor**

1. Definition of Capacitance

Consider two conductors carrying charges of equal magnitude and opposite sign. The quantity of charge Q on a capacitor is linearly proportional to the potential difference ΔV between the conductors of the capacitor

$$Q \;\alpha\; \Delta V$$

The proportionality constant depends on the **shape** and **separation** of the conductors. We can write this relationship as

$$Q = C\,\Delta V$$

The capacitance C of a capacitor is defined as the ratio of the magnitude of the charge on either conductor to the magnitude of the potential difference between the conductors:

$$C = \frac{Q}{\Delta V} \quad \textbf{Farad (F)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(1)}$$

We can calculate the capacitance for a **spherical charged** conductor, where the electric potential of the sphere of radius R is simply

$$\Delta V = KQ/R \qquad (\textit{Chapter 3}),$$

we have

$$C = \frac{Q}{\Delta V} \Rightarrow C = \frac{Q}{KQ/R} = \frac{R}{K}$$

$$C = 4\,\pi\,\varepsilon_0\,R \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(2)}$$

This expression shows that the capacitance of an isolated charged sphere is proportional to its radius and is independent of both the charge on the sphere and the potential difference. The capacitance of a pair of conductors depends on the geometry of the conductors as following.

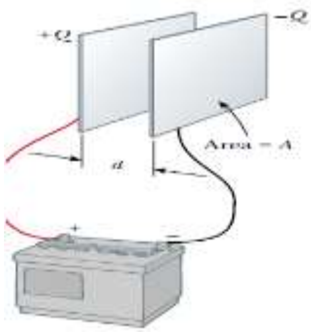
2. Calculating Capacitance

Let us illustrate this with three familiar geometries, namely, parallel plates, concentric Cylinders and concentric spheres.

A) Parallel-Plate Capacitors

Two parallel metallic plates of equal area A are separated by a distance d, as shown in Figure 1. One plate carries a charge + Q, and the other carries a charge -Q.

Fig.1 a



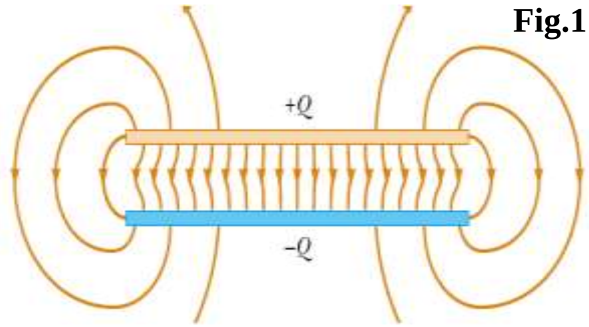


Fig.1 b

The value of the electric field between two parallel plates is

$$E = \frac{\sigma}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{Q}{\varepsilon_0 A}$$

Because the electric field between the plates of a parallel-plate capacitor is uniform near the center but no uniform near the edges (see **Fig.1**), the magnitude of the potential difference between the plates equals **Ed**; therefore,

$$\Delta V = Ed = \left(\frac{Q}{\varepsilon_0 A}\right) d$$

$$C = \frac{Q}{\Delta V} \quad \Rightarrow \quad C = \frac{Q}{Qd/\varepsilon_0 A}$$

$$C = \frac{\varepsilon_0 A}{d} \qquad \text{..............................} \qquad \textbf{(3)}$$

That is, the capacitance of a parallel-plate capacitor is proportional to the area of its plates and inversely proportional to the plate separation.

<u>Example 1</u>:

A parallel-plate capacitor with air between the plates has an area $A = 2 \times 10^{-4}$ m^2 and a plate separation d = 1mm. Find its capacitance.

<u>Solution</u>

$$C = \frac{\varepsilon_0 A}{d}$$

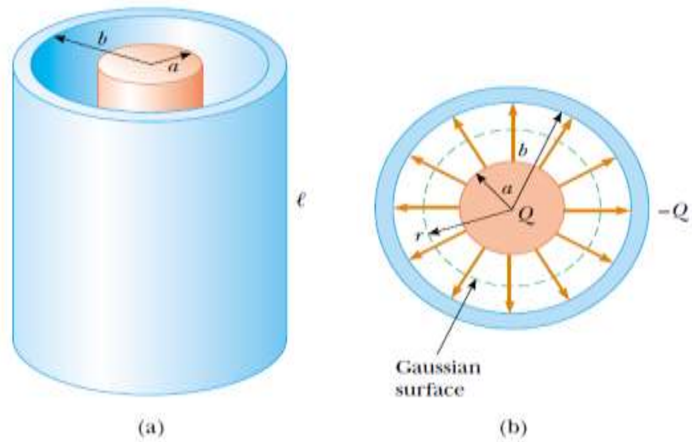
$$C = \frac{(8.85\times10^{-12})\,(2\times10^{-4})}{1\times10^{-3}} = 1.77\times10^{-12}\,F$$

$$C = 1.77 \ \text{pF}$$

B) The Cylindrical Capacitor

Consider a cylindrical conductor of length ℓ ,radius **a** and charge Q , is coaxial with a cylindrical shell radius b > a, and charge - Q (Figure 2). To find the capacitance of this cylindrical capacitor

Fig.2



The potential difference between the two cylinders, which is given in general by

$$V_b - V_a = -\int_a^b \overrightarrow{E} \cdot \overrightarrow{ds}$$

The electric field of a cylindrical charge distribution having linear charge density

$$E = 2K \ {}^{\lambda}\!/_{r}$$

$$V_b - V_a = -\int_a^b \overrightarrow{E_r} \cdot \overrightarrow{dr}$$

$$V_b - V_a = -2K\lambda \int_a^b \frac{dr}{r}$$

$$V_b - V_a = -2K\lambda \ln \left(\frac{b}{a}\right)$$

But $\lambda = {}^{Q}\!/_{\ell}$

$$C = \frac{Q}{\Delta V} = \frac{Q}{2K(Q/\ell)\ln\left(\frac{b}{a}\right)}$$

$$C = \frac{\ell}{2K \ln\left(\frac{b}{a}\right)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad \textbf{(4)}$$

C) The Spherical Capacitor

Consider a spherical capacitor consists radius b and charge - Q concentric with a smaller conducting sphere of radius **a** and charge +Q (Figure 3). To find the capacitance

$$V_b - V_a = -KQ\int_a^b \frac{dr}{r^2} \Rightarrow V_b - V_a = -KQ\int_a^b \frac{dr}{r^2}$$

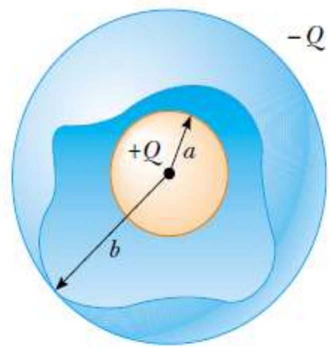


Fig.3

$$V_b - V_a = -KQ\left[\frac{1}{r}\right]_a^b$$

$$V_b - V_a = -KQ\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$V_b - V_a = KQ\,\frac{b-a}{ab}$$

But $C = \dfrac{Q}{\Delta V}$

$$C = \frac{ab}{K\,(b-a)} \qquad \text{..............................} \qquad \textbf{(5)}$$

Where b >> a

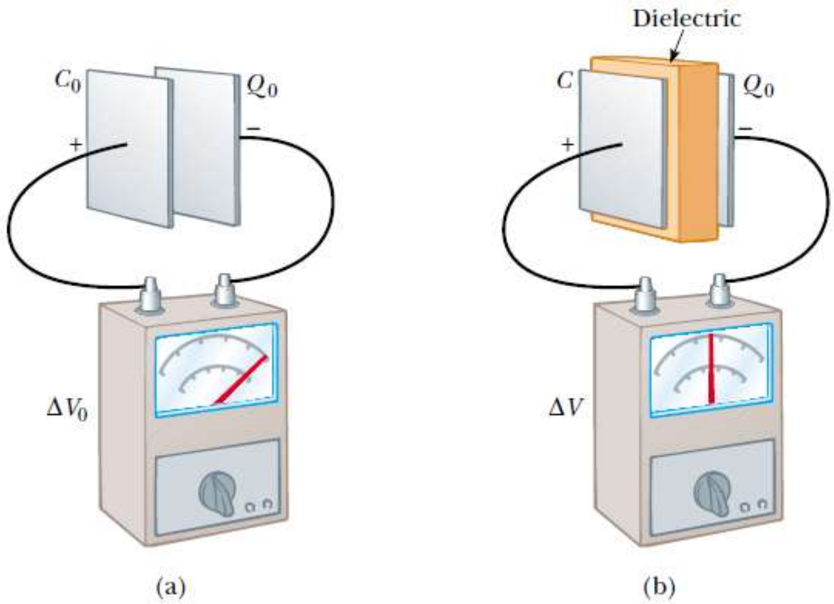
$$C = \lim_{b\to\infty}\frac{ab}{K\,(b-a)} \Rightarrow \frac{ab}{K\,(b)} = \frac{a}{K}$$

$$C = 4\pi\varepsilon_0 a \qquad \text{..............................} \qquad \textbf{(6)}$$

3) Capacitors with Dielectrics

A dielectric is a no conducting material, such as rubber, glass, or waxed paper. When a dielectric is inserted between the plates of a capacitor, then the capacitance increases. If the dielectric completely fills the space between the plates, the capacitance increases by a dimensionless factor ***K***, which is called the dielectric constant of the material. The dielectric constant varies from one material to another. A charged capacitor is (a) before and (b) after insertion of a dielectric between the plates, as shown in fig 4.

Fig.4



Consider a parallel-plate capacitor that without a dielectric has a charge Q_0 and a capacitance C_0. The potential difference across the capacitor is

$$\Delta V_0 = Q_0/C_0$$

The voltages with and without the dielectric are related by the factor ***K*** as follows:

$$\Delta V = \frac{\Delta V_0}{K}$$

Because K >1 and Q_0=Q (the charge Q_0 on the capacitor does not change),then $\Delta V < \Delta V_0$

$$C = \frac{Q_0}{\Delta V} \Rightarrow C = \frac{Q_0}{\Delta V_0 / K}$$

$$C = K \frac{Q_0}{\Delta V_0}$$

$$C = K C_0$$

The capacitance increases by the factor K when the dielectric completely fills the region between the plates.

$$C = K \frac{\varepsilon_0 A}{d} \quad \text{..............................} \quad \textbf{(7)}$$

If the dielectric is introduced while the potential difference is held constant by a battery, the charge increases to a value Q = KQ_0.

Example 2: A parallel-plate capacitor has plates of dimensions 2 cm by 3 cm separated by a 1 mm thickness of paper. Where K =3.7 and the dielectric strength of paper is 16 * 10^6 V/m.

(A) Find its capacitance. **(B)** What is the maximum charge that can be placed on the capacitor?

Solution

$$C = K\frac{\varepsilon_0 A}{d}$$

$$C = 3.7\frac{\left(8.85\times10^{-12}\right)\left(6\times10^{-4}\right)}{1\times10^{-3}} = 20\times10^{-12} \Rightarrow \quad C = 20\ \text{pF}$$

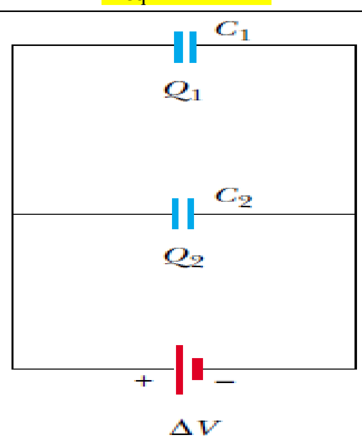
$$\Delta V_{max} = E_{max}d = \left(16\times10^{6}\right)\left(1\times10^{-3}\right) = 16\times10^{3}\ \text{V}$$

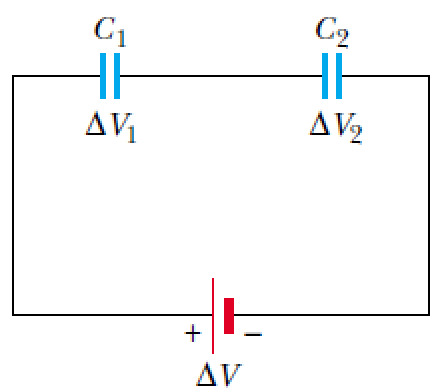
Hence, the maximum charge is

$$Q_{max} = C\Delta V_{max} = \left(20\times10^{-12}\right)\left(16\times10^{3}\right)\ \text{C} \qquad Q_{max} = 0.32\ \mu C$$

4. Combinations of Capacitors Parallel and Series Combination

In parallel combination	In series combination
the individual potential differences across capacitors connected are equal to the total potential difference $V_{total}=V_1=V_2$	The total potential difference across any number of capacitors is the sum of the potential differences across the individual capacitors. $V_{total}=V_1+V_2$
the total charge on capacitors is the sum of the charges on the individual capacitors $Q_{total} = Q_1+Q_2$	the charges on capacitors are the same $Q_{total} = Q_1=Q_2$
the equivalent capacitance of capacitors is the algebraic sum of the individual capacitances $C_{eq} = C_1+C_2$	the inverse of the equivalent capacitance is the algebraic sum of the inverses of the capacitances $1/C_{eq} = 1/C_1+1/C_2$

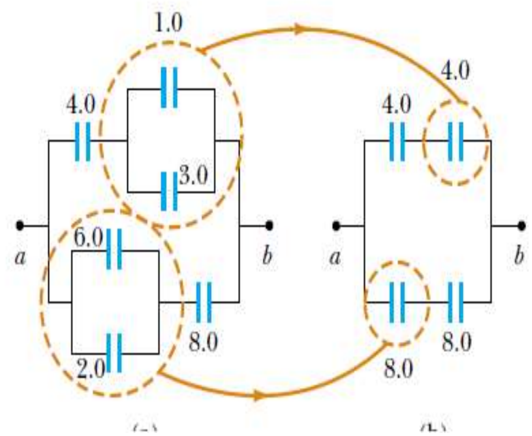




Example 3: Find the equivalent capacitance between a and b for the combination of capacitors shown in Figure 4. All capacitances are in microfarads.

Solution

Fig.4



$C_{eq1} = C_1 + C_2 =$ $C_{eq1} = 1\ \mu F + 3\ \mu F = 4\mu F$

$C_{eq2} = C_4 + C_5 =$ $C_{eq2} = 6\ \mu F + 2\ \mu F = 8\mu F$

$1/C_{eq3} = 1/C_3 + 1/C_{eq1} \Rightarrow 1/C_{eq3} = 1/4 + 1/4 \quad C_{eq3} = 2\ \mu F$

$1/C_{eq4} = 1/C_{eq2} + 1/C_6 \Rightarrow 1/C_{eq4} = 1/8 + 1/8 \quad C_{eq4} = 4\ \mu F$

$C_{total} = C_{eq3} + C_{eq4} \Rightarrow C_{total} = 2\mu F + 4\mu F = 6\ \mu F$

5) Energy Stored in a Charged Capacitor

Suppose that q is the charge on the capacitor at some instant during the charging process.

At the same instant, the potential difference across the capacitor is $\Delta V = \dfrac{q}{C}$

The work necessary to transfer an increment of charge **dq** from the plate carrying charge **-q** to the plate carrying charge **q** (which is at the higher electric potential) is

$$dW = \Delta V dq \Rightarrow dW = \frac{q}{C} dq$$

The total work required to charge the capacitor from q =0 to some final charge q = Q is

$$W = \int_0^Q \frac{q}{C} dq \Rightarrow W = \frac{1}{C}\int_0^Q q\ dq$$

$$W = \frac{Q^2}{2C} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(8)}$$

The work done in charging the capacitor appears as electric potential energy U stored in the capacitor. We can express the potential energy stored in a charged capacitor in the following forms:

$$U = \frac{Q^2}{2C} \Rightarrow U = \frac{1}{2} Q\ \Delta V$$

$$U = \frac{1}{2} C\ \Delta V^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad \mathbf{(9)}$$

This result applies to any capacitor, regardless of its geometry. Then the stored energy increases as the charge increases and as the potential difference increases.

We can consider the energy stored in a capacitor as being stored in the electric field created between the plates as the capacitor is charged. This description is reasonable because the electric field is proportional to the charge on the capacitor. For a parallel-plate capacitor, the potential difference and capacitance are equations

$$\Delta V = Ed. \quad \text{and} \quad C = \frac{\varepsilon_0 A}{d}$$

$$U = \frac{1}{2} \frac{\varepsilon_0 A}{d}\ E^2 d.^2$$

$$U = \frac{1}{2}(\varepsilon_0 A d) E^2$$

The energy per unit volume

$$u_E = \frac{U}{V} \qquad \text{Where V=Ad}$$

The energy density can be written

$$u_E = \frac{1}{2}(\varepsilon_0) E^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad \mathbf{(10)}$$

The energy density in any electric field is proportional to the square of the magnitude of the electric field at a given point.